

THE GROUP OF UNIMODULAR AUTOMORPHISMS OF \mathbb{C}^2 IS HOPFIAN

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ABSTRACT. Let G be the group of unimodular automorphisms of \mathbb{C}^2 . In the paper we prove two interesting results about this group. The first one is about absence of non-trivial finite-dimensional representations of G . The second one, we show that any non-trivial group endomorphism of G is a monomorphism, which implies that G is hopfian.

1. INTRODUCTION

Let $\text{Aut}(\mathbb{C}^2)$ be the group of polynomial automorphisms of the complex plane. Let G be the subgroup of automorphisms with Jacobian equal to 1. It is known that G can be written as the amalgamated product

$$(1) \quad G = A *_U B,$$

where A is the subgroup of symplectic affine automorphisms, B is the Jonquières subgroup

$$A = \{(ax + by + e, cx + dy + f)\}, \quad a, \dots, f \in \mathbb{C}, \quad ad - bc = 1$$

$$B = \{(ax + q(y), a^{-1}y + f)\}, \quad a \in \mathbb{C}^*, \quad f \in \mathbb{C}, \quad q(y) \in \mathbb{C}[y]$$

and $U = A \cap B$.

In [Sh], I. R. Shafarevich proved that G is simple as an infinite dimensional algebraic group. However, V. Danilov showed that G is not simple as an abstract group. In particular, he showed that there is an element of the *algebraic length* 26 (w.r.t. the above amalgamated product, see Section 2.1 for the precise definition), whose normal closure is not equal to G . Based on the work of Danilov, J.-P. Furter and S. Lamy [FL] showed that the normal closure of any element is non-trivial only if its length is at least 14 and equals to G if length is less or equal than 8. This is a main observation we use to show

Theorem 1. *There is no non-trivial finite dimensional representation of G .*

The group $\text{Aut}(\mathbb{C}^2)$ can be naturally embedded into $\text{Bir}(\mathbb{P}^2)$, and so does G . Recently, J. Déserti [De] has shown that any endomorphism of $\text{Bir}(\mathbb{P}^2)$ is injective. However this property is not functorial, therefore one can ask whether G is hopfian. Our main theorem

Theorem 2. *Any non-trivial endomorphism of G is injective. In particular G is hopfian, i.e. any epimorphism of G is an automorphism.*

Remark. In [W], D. Wright proved that $\text{Bir}(\mathbb{P}^2)$ can be presented as an amalgamation of three subgroups A_i ($i = 1, 2, 3$) along pairwise intersections. Moreover,

G can be embedded into $\text{Bir}(\mathbb{P}^2)$ via inclusions of A and B into A_2 and A_3 respectively (see, *loc.cit.*, Theorem 3.13 and Theorem 4.21). In light of this, it would be interesting to see the relation between our Theorem 2 and Déserti's result.

The paper is organized as follows. In section 2 we recall some facts and results needed in later sections. In section 3 we prove Theorem 1. In section 4 we prove Theorem 2.

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2. PRELIMINARIES

2.1. On subgroups of G . By [FM], the elements of G can be divided into two separate classes according to their dynamical properties as automorphisms of \mathbb{C}^2 : every $g \in G$ is conjugate to either an element of B or a composition of generalized Hénon automorphisms of the form:

$$\sigma g \sigma^{-1} = g_1 g_2 \dots g_m ,$$

where $g_i = (y, x + q_i(y))$ with polynomials $q_i(y) \in \mathbb{C}[y]$ of degree ≥ 2 . We say that g is the *elementary* or *Hénon* type, respectively. A subgroup $H \subseteq G$ is called the *elementary* if each element of H is of elementary type.

The following results are proved in [L, Theorem 2.4, Proposition 4.8]

Theorem 3. (a) *Let H be an elementary subgroup of G . Then one of the following occurs :*

- (1) *H is either conjugate to A or B .*
- (2) *H is not conjugate to A or B . Then H is abelian.*
- (b) *Let $g \in G$ be an element of Hénon type. Then its centralizer is isomorphic to $\mathbb{Z} \rtimes \mathbb{Z}/p\mathbb{Z}$. In particular it is countable.*

Remark. It is easy to see that the centralizer of any automorphism of the elementary type is uncountable.

Let $g \in G$ be an element which is not in U . Then we say that it has the *algebraic length* m , if m is the least integer such that $g = \sigma_1 \dots \sigma_m$ and each g_i is either in A or in B . We denote $|g| = m$. If $g \in U$ then we define $|g| = 0$. The following is proved in [FL, Theorem 1]

Theorem 4. *If $g \in G$ satisfies $|g| \leq 8$ and $g \neq \text{Id}$, then the normal subgroup generated by g is G .*

2.2. Divisible groups. We review some facts and notions about divisible groups.

We recall that an abelian group H is *divisible* if for each $g \in H$ and positive integer n there is an element $h \in H$ with $g = nh$.

Finite abelian groups are not divisible. Among familiar infinite abelian groups, $\mathbb{Q}, \mathbb{R}, \mathbb{C}, \mathbb{C}[y], \mathbb{C}^*$ are divisible but \mathbb{R}^* and \mathbb{Z} are not.

The following fact is useful

Lemma 1. *A quotient of a divisible group is divisible.*

In particular we have

Corollary 1. *There are no non-trivial homomorphisms from divisible groups to finite groups.*

2.3. Solvable subgroups of $\mathrm{GL}_n(\mathbb{C})$. We recall a classical characterization of solvable subgroups of $\mathrm{GL}_n(\mathbb{C})$ due to A. I. Maltsev. Maltsev's theorem is a generalization of the Lie-Kolchin theorem, it gives a description of all solvable subgroups of $\mathrm{GL}_n(\mathbb{C})$ for its proof we refer to [LR, Theorem 3.1.6].

Theorem 5 (Maltsev). *Let Γ be any solvable subgroup of $\mathrm{GL}_n(\mathbb{C})$. Then Γ has a finite index normal subgroup which is conjugate to a subgroup of upper triangular matrices.*

Let U_n be the group of upper triangular matrices with entries 1 in the diagonal. One can easily show

Lemma 2. *U_n is a subgroup of $\mathrm{GL}_n(\mathbb{C})$ of nilpotency class $n - 1$.*

3. ON NONEXISTENCE OF FINITE-DIMENSIONAL REPRESENTATIONS OF G

Let $\rho : G \rightarrow \mathrm{GL}_n(\mathbb{C})$ be a group homomorphism. Since $G = [G, G]$ we can easily see that $\rho(G) \subseteq \mathrm{SL}_n(\mathbb{C})$. Now we prove

Proposition 1. *If ρ is non-trivial then $\rho|_A$ and $\rho|_B$ must be injective.*

Proof. Suppose the kernel of ρ contains g , a non-trivial element of A or B . Since g is of length at most one, by Theorem 4 the normal closure of any such element is equal to G . This implies that ρ must be trivial. \square

Since B is a solvable group, $\rho(B)$ is a solvable subgroup of $\mathrm{SL}_n(\mathbb{C})$. The following lemma gives more precise description of $\rho(B)$:

Lemma 3. *All eigenvalues of $\rho(x + \lambda y^k, y)$ and $\rho(x, y + \mu x^k)$ are 1 for all $\lambda, \mu \in \mathbb{C}$ and $k \geq 0$.*

Proof. We set $A_{k,\lambda} := \rho(x + \lambda y^k, y)$ and $B_{k,\mu} := \rho(x, y + \mu x^k)$. We will prove the lemma for $A_{k,\lambda}$ since a proof of $B_{k,\mu}$ is analogous. We have

$$(\nu^{-1}x, \nu y) \circ (x + \lambda y^k, y) \circ (\nu x, \nu^{-1}y) = (x + \lambda \nu^{k+1}y^k, y).$$

for some $\nu \in \mathbb{C}^*$. This means a matrix $A_{k,\lambda}$ is similar to $A_{k,\lambda\nu^{k+1}}$. In particular for $\nu^{k+1} \in \mathbb{Z}$ we obtain that $A_{k,\lambda}$ is similar to any of its power $A_{k,m\lambda} = A_{k,\lambda}^m$. If $\{a_1, \dots, a_n\}$ is the set of eigenvalues of $A_{k,\lambda}$, then this set equal to the set $\{a_1^m, \dots, a_n^m\}$ for any $m \geq 1$. This implies that $a_1^{m_1} = 1, \dots, a_n^{m_n} = 1$ for some positive m_1, m_2, \dots, m_n . Finally, choosing $m = m_1 m_2 \dots m_n$ we have $\{a_1^m = \dots = a_n^m = 1\}$. Hence $a_1 = a_2 = \dots = a_n = 1$. \square

Consider the unitriangular subgroup $B_0 \subseteq B$ consisting of elements

$$(x + p(y), y + f).$$

Then we have

Proposition 2. *$\rho(B_0)$ is conjugate to a subgroup of U_n .*

Proof. First we note B_0 is a solvable subgroup B and $\rho(B_0)$ is a solvable subgroup of $\mathrm{SL}_n(\mathbb{C})$. Therefore by Theorem 5 it has a normal triangularizable subgroup T which has a finite index in $\rho(B_0)$. In other words $\rho(B_0)/T$ is a finite group. A surjective homomorphism $B_0 \rightarrow \rho(B_0)/T$ induces a homomorphism $[B_0, B_0] \rightarrow \rho(B_0)/T$. Since a group $[B_0, B_0] \cong \mathbb{C}[y]$ is divisible, $[B_0, B_0] \rightarrow \rho(B_0)/T$ must be trivial. Therefore we have a surjective homomorphism $B_0/[B_0, B_0] \rightarrow \rho(B_0)/T$. However $B_0/[B_0, B_0] \cong \mathbb{C}$ is divisible therefore a group $\rho(B_0)/T$ must be trivial. Hence $\rho(B_0)$ is conjugate to a subgroup of upper triangular matrices. Now by Lemma 3 $\rho(B_0)$ is also unipotent. \square

On the other hand

Proposition 3. *B_0 is not a nilpotent group.*

Proof. One can compute that the group $B_0^{(1)} = [B_0, B_0]$ consists of elements

$$(x + p(y), y) \quad \text{for all } p(y) \in \mathbb{C}[y]$$

On the other hand

$$B_0^{(2)} = [B_0, B_0^{(1)}] = B_0^{(1)}$$

So it stabilizes $1 \neq B_0^{(1)} = B_0^{(2)} = \dots$. Hence it is not nilpotent. \square

Proof of Theorem 1. Suppose there is a non-trivial homomorphism $\rho : G \rightarrow \mathrm{GL}_n(\mathbb{C})$. By Proposition 1 its restriction to B_0 must be injective. From Proposition 2 it follows that $\rho(B_0)$ can be conjugated to a subgroup of U_n and hence is nilpotent. This contradicts to Proposition 3. \square

There are some interesting consequences of this result which are of independent interest. Let $\mathrm{Cr}(n)$ be the Cremona group of birational automorphisms of \mathbb{P}^n . Then the above result implies

Corollary 2. (a) *There is no non-trivial finite dimensional representation of $\mathrm{Cr}(2)$.*
(b) *$\mathrm{Aut}(\mathbb{C}^n)$ and $\mathrm{Cr}(n)$ are not linear, i.e., these groups have no faithful representations in $\mathrm{GL}_n(\mathbb{C})$.*

Proof. (a) Follows from the fact that the subgroup $\mathrm{SL}_2(\mathbb{C})$ in G is also a subgroup of $\mathrm{PGL}_3(\mathbb{C})$ in $\mathrm{Cr}(2)$.

(b) It follows immediately from the fact that G is a subgroup of both $\mathrm{Aut}(\mathbb{C}^n)$ and $\mathrm{Cr}(n)$. \square

Results of this corollary for $\mathrm{Cr}(n)$ were proved earlier by D.Cerveau and J.Déserti [CD].

4. ENDOMORPHISMS OF THE GROUP G

For $g \in G$ we denote by Ad_g the inner automorphism of G given by $g(\cdot)g^{-1}$. To prove our theorem it suffices to show: given a non-trivial $\phi : G \rightarrow G$ homomorphism there are $g, h \in G$ such that composition $\mathrm{Ad}_g \circ \phi \circ \mathrm{Ad}_h$ is a monomorphism. First we will show that any non-trivial endomorphism of G can be composed by an inner automorphism to give an endomorphism which induces injective endomorphisms of its subgroups A and B , namely $\phi(A) \subset A$ and $\phi(B) \subset B$. Following [FL] one can define systems of representatives of the non-trivial left cosets A/U and B/U by

$$I = \{ (\lambda x + y, -x), \lambda \in \mathbb{C} \}$$

$$J = \{ (x + p(y), y), p(y) \in y^2\mathbb{C}[y] \setminus \{0\} \}$$

respectively. We can prove

Proposition 4. *Let $\mu = (x + p(y), y)$ such that $\deg(p) = n \geq 2$. Then $A \cap \mu A \mu^{-1}$ is a subgroup of H defined as*

$$(2) \quad H = \{ (x + by + e, y) \mid b, e \in \mathbb{C} \} \rtimes \mathbb{Z}_{n+1}$$

where \mathbb{Z}_{n+1} is the cyclic subgroup of $(\lambda x, \lambda^{-1}y)$, $\lambda \in \mathbb{C}^*$.

Proof. We will consider two cases: $g \in A \setminus U$ and $g \in U$. In the first case $\mu g \mu^{-1}$ is a word of length 3 so it can not be in A . If $g \in U$ we have

$$\mu g \mu^{-1} = (\lambda x + \lambda p(y) - p(\lambda^{-1}y + f) + by + e, \lambda^{-1}y + f)$$

where $g = (\lambda x + by + e, \lambda^{-1}y + f)$. The element $\mu g \mu^{-1}$ belongs to A if and only if $\deg(\lambda p(y) - p(\lambda^{-1}y + f)) \leq 1$ which can only happen if $\lambda^{n+1} = 1$. This immediately imply the statement. \square

Proposition 5. *Let $\phi : G \rightarrow G$ be a non-trivial group homomorphism. Then*

- (a) *Restrictions of ϕ to A and B are group monomorphisms.*
- (b) *$\phi(A) \cap \phi(B) = \phi(U)$.*

Proof.

- (a) Let $a \in A \cup B$ be an element $a \neq 1$ such that $\phi(a) = 1$. Then by Theorem 4 we have $\phi(G) = 1$, which is impossible.
- (b) It is clear that $\phi(U) \subset \phi(A) \cap \phi(B)$. Now if $\phi(a) = \phi(b)$ for some $a \in A$ and $b \in B$ then $\phi(ab^{-1}) = 1$. Again by Theorem 4 implies that ϕ is injective on words of length 2. Hence $a = b \in U$.

\square

Theorem 6. *Let $\phi : G \rightarrow G$ be a non-trivial group homomorphism. Then composing ϕ by proper inner automorphisms of G , we obtain a homomorphism $\tilde{\psi}$ such that*

$$\tilde{\psi}(A) \subset A, \tilde{\psi}(B) \subset B, \tilde{\psi}(U) \subset U.$$

Moreover ψ restricted to A, B and U gives injective endomorphisms.

Proof. By Theorem 3(b) each element of subgroups $\phi(A)$ and $\phi(B)$ is elementary. Hence, by part (a) of the same theorem both $\phi(A)$ and $\phi(B)$ can be conjugated to either A or B . The subgroup $\phi(A)$ can not be conjugated to B , since B is solvable while $\phi(A)$, being isomorphic to A , is not. So $\phi(A) \subset \sigma A \sigma^{-1}$ for some $\sigma \in G$. Composing ϕ by $\text{Ad}_{\sigma^{-1}}$ we can assume that $\phi(A) \subset A$.

For $\phi(B)$ we have that it is conjugate to A or B . We now discuss each case.

Case 1. Assume that $\phi(B) \subset \mu A \mu^{-1}$ for some $\mu \in G$. Let μ be of length 1. If $\mu \in A$ then $\phi(B) \subset A$ and this implies $\phi(G) \subset A$. Taking projection of A onto $\text{SL}_2(\mathbb{C})$ we get a representation of G which by Theorem 1 is trivial. Therefore we obtain a homomorphism $G \rightarrow \mathcal{T}$, where $\mathcal{T} = \{(x+e, y+f) \mid e, f \in \mathbb{C}\}$ the translation subgroup, which must be injective when restricted to A and B by Proposition 5. This is impossible, hence $\mu \notin A$.

Now assume that $\mu \in B$. Without loss of generality we can assume μ is a non-trivial representative in B/U , with $\mu = (x + p(y), y)$ with $p \in y^2\mathbb{C}[y] \setminus 0$. Then by Proposition 5(b) we have $\phi(U) \subseteq A \cap \mu A \mu^{-1}$. Then according to Proposition 4 the

group U embeds into (2). Note that U contains a cyclic group of any finite order, which contradicts to the last embedding. Hence $\mu \notin B$.

Let

$$(3) \quad \mu = w_0 w_1 \dots w_n, \quad n \geq 2$$

be a reduced word of length n in G where $w_0 \in U$ and w_i for $i > 0$ are in I or J . Without loss of generality we can assume $w_n \in J$. Once again by Proposition 5(b) we must have $\phi(U) \subseteq A \cap \mu A \mu^{-1}$. Then

$$\mu a \mu^{-1} = w_0 w_1 \dots w_n a w_n^{-1} \dots w_1^{-1} w_0^{-1},$$

and $\mu a \mu^{-1} \in A$ if and only if

$$\nu = w_0^{-1} \mu a \mu^{-1} w_0 = w_1 \dots w_n a w_n^{-1} \dots w_1^{-1}.$$

is in A . Now either $a \in A \setminus U$ or $a \in U$. In the first case ν is a word of length $2n + 1$ so it can not be in A . If $a \in U$ then the element $w_n a w_n^{-1}$ is in U or $B \setminus U$. In the latter case ν is at least of length $2n - 1 > 2$ and therefore it is not in A . If $w_n a w_n^{-1}$ is in U then all such elements are in $A \cap w_n A w_n^{-1}$, i.e. $A \cap \mu A \mu^{-1}$ can be embedded in $A \cap w_n A w_n^{-1}$. By Proposition 4 $A \cap w_n A w_n^{-1}$ is a subgroup of H and hence $A \cap \mu A \mu^{-1}$ can be embedded into H . By injectivity of ϕ on U , this is impossible since U contains a cyclic group of any finite order. Thus $\phi(B)$ can not be conjugated to a subgroup of A .

Case 2. Let $\phi(B) \subset \mu B \mu^{-1}$ for some $\mu \in G$. Then $\phi(U) \subset A \cap \mu B \mu^{-1}$. Now if μ is in A or B then we are done. So we assume that μ has a reduced form as in (3). Arguing as above we can show that U can not be isomorphic to a subgroup of $A \cap \mu B \mu^{-1}$.

Thus summarizing all cases we conclude that by composing ϕ by an inner automorphism of A if necessary, we obtain a homomorphism $\tilde{\psi}$ with properties stated in the theorem. \square

We can slightly refine the previous theorem

Lemma 4. *Let $\tilde{\psi}$ be as in Theorem 6. Then composing $\tilde{\psi}$ by inner automorphisms Ad_g with $g \in U$ we obtain ψ such that*

$$\psi(A) \subset A, \psi(B) \subset B, \psi(U) \subset U$$

and

$$\psi((-x, -y)) = (-x, -y)$$

Proof. By Theorem 6 $\tilde{\psi}(U) \subset U$ therefore

$$\tilde{\psi}((-x, -y)) = (\lambda x + cy + e, \lambda^{-1}y + f) \quad \text{for some } \lambda \in \mathbb{C}^*, c, e, f \in \mathbb{C}$$

Since $(-x, -y)$ is of order 2 and ψ is injective on U we have $\tilde{\psi}((-x, -y)) = (-x + e, -y + f)$. Now if we take $g = (\frac{e}{2}, \frac{f}{2})$, the composition $\psi = \text{Ad}_g \circ \tilde{\psi}$ gives us desired homomorphism. \square

In particular we have

Corollary 3. *Let ψ be as in Lemma 4. Then $\psi(\text{SL}_2(\mathbb{C})) \subset \text{SL}_2(\mathbb{C})$.*

Proof. Let $Z_A(g)$ be the centralizer subgroup of g in A . Then $\psi(Z_A(g)) \subseteq Z_A(\psi(g))$. Now proof follows from $Z_A((-x, -y)) = \text{SL}_2(\mathbb{C})$. \square

5. THE PROOF OF THEOREM 2

We need to show that a homomorphism ψ with properties described in Theorem 6 and Lemma 4 is injective. By Theorem 6 and Lemma 4 for ψ we have induced quotient maps

$$\bar{\psi}_A : A/U \rightarrow A/U, \quad \bar{\psi}_B : B/U \rightarrow B/U$$

To prove our result it is sufficient to show that these maps are injective. Indeed, assume that these two maps are injective and let $g \in G$ be $g \neq 1$. It has a normal form $g = w_0 w_1 \dots w_n \neq 1$ where $w_0 \in U$ and w_i are in I or J . Then

$$\psi(g) = \psi(w_0) \psi(w_1) \dots \psi(w_n),$$

where $\psi(w_0) \in U$ and $\psi(w_i)$ are non-trivial representatives in A/U or B/U by injectivity of $\bar{\psi}_A$ and $\bar{\psi}_B$. So the above presentation of $\psi(g)$ is a reduced word and can not be equal to 1.

Proof of injectivity of $\bar{\psi}_A$: Recall A/U consists of gU , for $g \in I$. Suppose $\psi(g) \in U$ for some $g \in I$. Since g and U generate A , we have $\psi(A) \subset U$. The latter contradicts injectivity of $\psi|_A$ since U is solvable while A is not. Therefore $\bar{\psi}_A$ must be injective.

Proof of injectivity of $\bar{\psi}_B$: Coset representatives B/U consists of gU , where $g \in J$. Suppose that $\bar{\psi}_B$ is not injective, namely there is $g = (x + p(y), y)$ with nonzero $p(y) \in y^2 \mathbb{C}[y]$ such that $\psi((x + p(y), y)) \in U$. Let $\deg(p(y)) = n > 1$. Then the image of the following is also in U

$$[(x, y + 1), (x + p(y), y)] = (x + p(y + 1) - p(y), y).$$

Note that the degree of $p(y + 1) - p(y)$ is exactly $n - 1$ and taking commutator with $(x, y + 1)$ lowers degree exactly by 1. Therefore taking commutator $(x + p(y), y)$ with $(x, y + 1)$ exactly $n - 2$ times gives us

$$[(x, y + 1), [(x, y + 1), \dots, [(x, y + 1), (x + p(y), y)] \dots]] = (x + q(y), y),$$

where $q(y)$ is a quadratic polynomial and $\psi((x + q(y), y)) \in U$. Therefore $\psi((x + y^2, y))$ is also in U . Note since $\psi(B) \subset B$ we also have $\psi(B^{(i)}) \subset B^{(i)}$ for derived series of B . In particular since $B^{(2)} = \{(x + p(y), y)\}$ we obtain that $\psi((x + y^2, y)) = (x + cy + e, y)$ for some $c, e \in \mathbb{C}$. We have

$$[(-x, -y), (x + y^2, y)] = (x - 2y^2, y) \quad \text{and} \quad [(-x, -y), (x + cy + e, y)] = (x - 2e, y)$$

Therefore by Lemma 4 we have $\psi((x - 2y^2, y)) = (x - 2e, y)$. On the other hand,

$$\psi((x - 2y^2, y)) = \psi((x + y^2, y)^{-2}) = (x + cy + e, y)^{-2} = (x - 2cy - 2e, y).$$

Hence $c = 0$ and therefore $\psi((x + y^2, y)) \in \mathcal{T}$. Note then by Corollary 3 the element $\psi((-y, x)(x + y^2, y)(y, -x))$ is also in \mathcal{T} . Therefore ψ maps commutator of $(x + y^2, y)$ and $(-y, x)(x + y^2, y)(y, -x)$ which is of length 8 to identity. This is impossible by Theorem 4. This completes a proof of injectivity of ψ_B hence of ψ .

Remark. One can prove using similar arguments that $\text{Aut}(\mathbb{C}^2)$ is also hopfian. However one can easily observe not every endomorphism of $\text{Aut}(\mathbb{C}^2)$ is injective.

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